Misinterpretations of Bernoulli's Law

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1. Common derivation and applications of Bernoulli's law

In a recent paper Baumann and Schwaneberg (1994) state:

> *Bernoulli's Equation is one of the more popular topics in elementary physics. It provides striking lecture demonstrations, challenging practice problems, and plentiful examples of practical applications from curving baseballs to aerodynamic lift. Nevertheless, Students and Instructors are often left with an uncomfortable feeling that the equation is clear and its predictions are verified, but the real underlying cause of the predicted pressure changes is obscure.*

This statement is correct but it must be added that the common treatment of Bernoulli's equation is misleading. Generally the basic law of conservation of energy is applied to the flow of an incompressible fluid through a tube with different cross-sections.

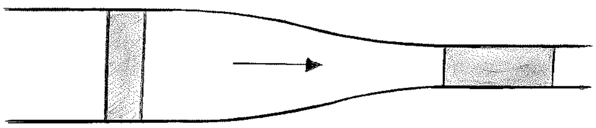

Figure 1

The energy of a volume V at any point is the sum of its kinetic energy $\left(\frac{m}{2}v^2\right)$ and its potential energy (pV).

Effects of gravitation and viscosity are neglected. The energy of a given volume of the fluid which moves from point 1 to point 2 is the same at both points. The related energy equation is

$$p_1 V + \frac{m}{2}v_1^2 = p_2 V + \frac{m}{2}v_2^2 \qquad (1)$$

Using $m = \rho V$ and rearranging we arrive at Bernoulli's Law:

$$p_1 - p_2 = \frac{\rho}{2}\left(v_2^2 - v_1^2\right) \qquad (2)$$

The equation states a reversed relation between static pressure and streaming velocity which is often demonstrated by experiments like

- *Soaring ball*: A light ball (e.g. ping pong ball) can be kept soaring in an upwards directed air stream of a hair dryer. The ball remains within the stream even if the stream is inclined and not vertical. The explanation given is that the static pressure within the stream is less due to the higher velocity.

- *Evaporator*: If a fast stream of air passes over the opening of a pipe, the pressure inside is lowered and it is possible to suck in liquids. This effect is used as an application of Bernoulli's law (Paus 1995) referring to the high streaming velocity within the air stream and claiming $p_{stream} < p_{atmosphere}$.

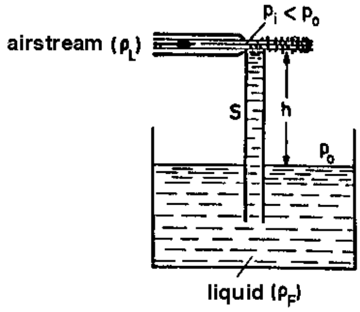

Figure 2

- *Aerodynamic lift:* The higher streaming velocity of the air at the upper surface of the wing is stated to be the cause of the lower pressure. Different reasons are given for the generation of the higher streaming velocity. The most popular one is a comparison of path lengths of the flow above and below the aerofoil and the statement that due to a longer path length at the upper side the flow has to be faster (Mansfield and Sullivan 1998, Baumann and Schwaneberg 1994) .

2. Misinterpretations and misapplications of Bernoulli's law

2.1 Static pressure in a free air stream

Static pressure is the pressure inside the stream measured by a manometer moving with the flow. At the same time, the static pressure is the pressure which is excerted on a plane parallel to the flow. Thus the static pressure within an air stream has to be measured carefully using a special probe. A thin disk must cover the probe except for the opening. The disk must be positioned parallel to the streaming flow, so that the flow is not interfered with. A sufficiently sensitive manometer can be produced easily if not available in the lab. A fine pipe of glass is bent at one side to dip in a cup and to be fixed according to figure 3. The meniscus must be positioned in the middle of the pipe. The suitable inclination should be 1:15 - 1:30. A rubber tube connects the glass pipe with a probe. A flat disk must be glued on top of the probe leaving the opening free. The disk has to be held parallel to the streaming.

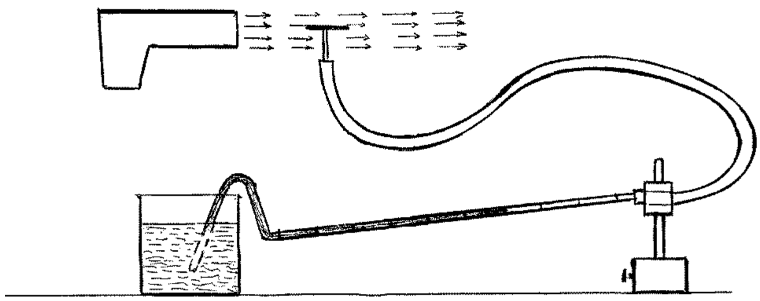

Figure 3

If the static pressure is measured in the way outlined above within a free air stream generated by a fan or a hair dryer it can be shown that the static pressure is the same as in the surrounding atmosphere. Bernoulli's law cannot be applied to a free air stream because friction plays an important role. It may be noted that the situation is similar to the laminar flow of a liquid with viscosity inside a tube. The different velocity of the stream layers is caused by viscosity. The static pressure is the same throughout the whole cross-section. A free air stream in the atmosphere is decelerated by friction. If static pressure in a free air stream is equal to atmospheric pressure, some of the striking lecture demonstrations are interpreted incorrectly since the effects observed are not caused by Bernoulli's law.

2.2. Aerodynamic lift

The explications referring to differences of path lengths are wrong. Air volumes which are adjacent before separation at the leading edge of the aerofoil do not meet again at the trailing edge (Weltner 2010). Thee higher streaming velocity at the upper surface of the aerofoil is not the cause of lower pressure. It is the other way round as will be shown below. The the higher streaming velocity is the consequence of the lower pressure at the upper surface of the wing (Weltner 2010).

These contradictions and misunderstandings can only be clarified by means of the basic physics of fluid mechanics and we cannot avoid some calculations.

3. Fluid dynamics, Newtons laws and the Euler equations

Fluid dynamic is an extension of Newton`s mechanic. It was Euler who applied the fundamental laws of Newton to fluid motion. For simplicity reasons we restrict our considerations to stationary flow and we neglect effects of gravitation and viscosity. We assume a cubical volume with mass Δm. (Figure 4). The basic equation of motion is:

$$F = \Delta m a. \tag{3}$$

We analyse separately *tangential acceleration*, figure 4 and *normal acceleration*, figure 5.

Tangential acceleration

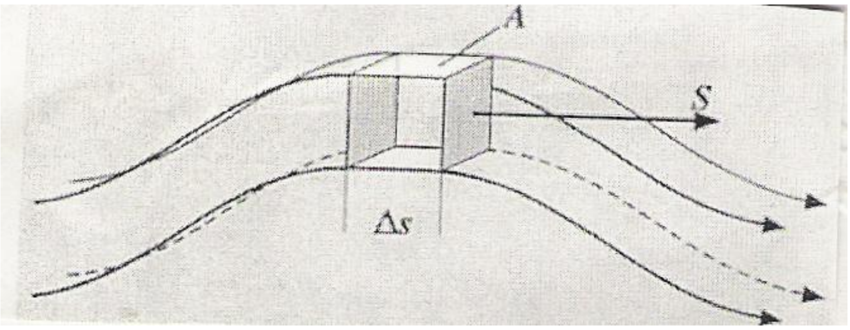

Figure 4: Tangential acceleration of a cubical volume within curved stream lines.

A tangential acceleration in s-direction is the result of a force. A force in s-direction occurs when the pressure acting at the face A at the back is higher than the pressure in front. An acceleration in s-direction is caused by a decrease of pressure in s-direction.

$$F = \Delta m \cdot \ddot{s} = -A\frac{\delta p}{\delta s} \cdot \Delta s. \qquad (4)$$

Inserting the mass $\Delta m = \rho A \Delta m$ and $\ddot{s} = \frac{dv}{dt}$ we get

$$\rho\frac{dv}{dt} = -\frac{dp}{ds}. \qquad (5)$$

This equation is transformed to

$$\int_1^2 \rho \cdot dv\frac{ds}{dt} = -\int_1^2 dp. \qquad (6)$$

Solving the definite integral we arrive at the Bernoulli equation:

$$\frac{\rho}{2}\left(v_2^2 - v_1^2\right) = p_1 - p_2 .$$

This derivation is not as simple as the derivation based on energy conservation cited in the beginning, but it has a great advantage. It shows clearly that higher streaming velocity at a given point is a consequence of a lower pressure at this point.

Normal acceleration

(See figure 5). A *normal acceleration* within curved streamlines needs a higher pressure at the outer lateral face A than at the inner lateral face.

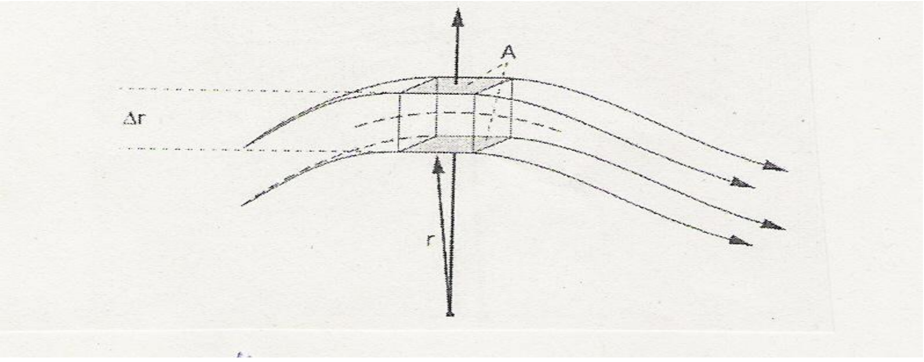

Figure 5: Normal acceleration of a volume element within curved streamlines.

According to figure 5: $F = -A\dfrac{dp}{dz}\cdot \Delta z = \Delta m \cdot \ddot{z}\,.$ (8)

Inserting the mass of the volume $\Delta m = \rho A \cdot \Delta z$ we arrive at

$$\frac{dp}{dz} = -\rho \cdot \ddot{z}\,. \qquad (9)$$

The acceleration in direction of the centre of curvature is well known. It is the centripetal acceleration of a circular motion.

$\ddot{z} = -\dfrac{v^2}{R}$ (R = radius of curvature, v = streaming velocity). (10)

Finally we obtain

$$\frac{dp}{dz} = \rho\frac{v^2}{R}\,. \qquad (11)$$

Curved streamlines within a flow are related to pressure gradients. Unfortunately this equation cannot be integrated directly. The integration requires the knowledge of the total flow field. However this relation can be demonstrated in a simple and impressing way. If we make water rotate in a pot we generate a circular streaming flow. In this case the surface of the water rises at the outer parts. The level of the water is a manometer indicating the pressure beneath thus indicating the growing pressure with growing radius. This effect explains the function of centrifuges as well..

As a rule, physics textbooks neglect the treatment of normal acceleration of fluids in curved streamlines. Thus they omit to discuss the pressure gradients normal to the velocity if streamlines are curved. By the way, this is different from textbooks on technical fluid dynamics which treat the flow of fluids in curved tubes. The neglect of pressure gradients related to curved streamlines is disastrous because the mechanism producing low pressure in streaming fluids is thus made impossible to be understood.

Obstacles cause curved streamlines and generate pressure gradients of air and thus cause regions of higher or lower pressure. The deflection of the streaming fluid is the cause for the generation of pressure gradients perpendicular to the streamlines and thus causes regions of higher or lower pressure.

The flow near limiting surfaces follows the geometrical shape of these surfaces. This behaviour is called Coanda-effect. It is neither trivial nor general. The flow must not be forced to change its direction abruptly as to avoid the generation of turbulence and separation. The Coanda-effect holds for all flows limited by smoothly curved surfaces like aerofoils, streamlined obstacles, sails and –with a certain reservation- roofs. This effect can be understood taking viscosity into consideration. In figure 7 we assume a stream to start. It will flow horizontally. But due to viscosity some layers of the adjacent air will be taken away by the stream. In this region –dotted in figure 7- the air is sucked away and hence gives rise to a reduction of pressure and thus generating a normal acceleration of the stream. By the end of this process the stream fits the shape of the curved surface, figure 6.

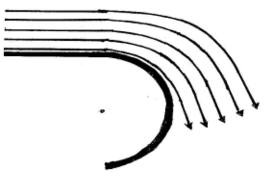

Figure 6

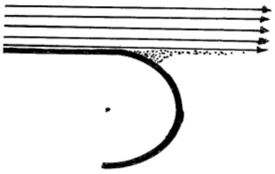

Figure 7

This *Gedankenversuch* illustrates the importance of viscosity in generating a stationary flow.

4. Generation of high and low pressure within a flow

4.1 Measurement of static pressure within a free stream of air

According to figure 8 we place a curved plane into the air stream of a fan. A curved plane can be produced by glueing two postcards on top of each other. By fixing them around a bottle with a rubber, an appropriate curvature can be achieved. Due to the Coanda-effect, the air stream follows the shape of the curved plane at both sides.
The curved plane generates a curved flow resulting in a radial pressure gradient at both sides. Outside of the flow there is atmospheric pressure.
In relation to the center of curvature the pressure increases if we go from the center away.

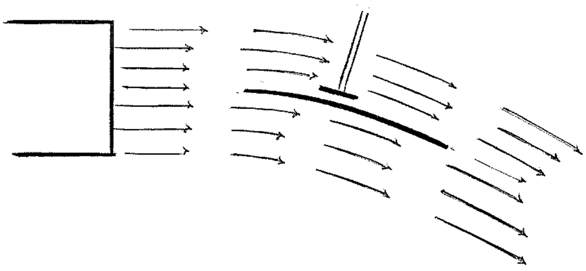

Figure 8

Let us first regard the upper convex surface. Going away from the center of curvature the pressure increases and in greater distance it will be atmospheric pressure. Thus there must be less pressure at the surface compared to atmospheric pressure.

Now we regard the inner or concave side, figure 9. Going away from the surface a decrease of pressure is to be expected. Thus we have higher pressure at the surface compared to atmospheric pressure.

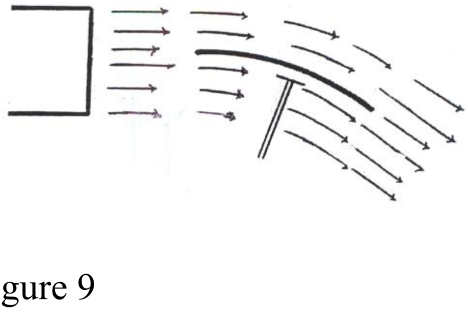

Figure 9

This can be demonstrated using the manometer described at the beginning. The experiment shows that by deflecting of an air stream regions of increased or decreased pressure are generated. This experiment is fundamental for the understanding of the production of pressure differences if air passes obstacles. By analysing the curvature of the evasive flow we can predict pressure distribution. It should be added that in this case Bernoulli's law still holds. Since in figure 8 the pressure at the convex surface decreases the local streaming velocity is increased because the incoming air is accelerated.

In figure 9 the pressure increases at the concave surface and the streaming velocity decreases because the incoming air will be decelerated.

The experiment requires an air stream the cross section of which should exceed the width of the postcard. If a hair-dryer is used which produces a narrow air stream it is advisable to glue the curved plane between two even planes of glass or plastic to confine the air stream. The distance of the limiting planes should be equal to the diameter of the air stream produced by the hair-dryer.

4.2 Examples and applications

Hill: If air passes a hill - figure 10- it follows the shape of it. A deformation of the original horizontal flow occurs only in the surroundings of the hill. Further away we observe normal atmospheric pressure and horizontal flow.

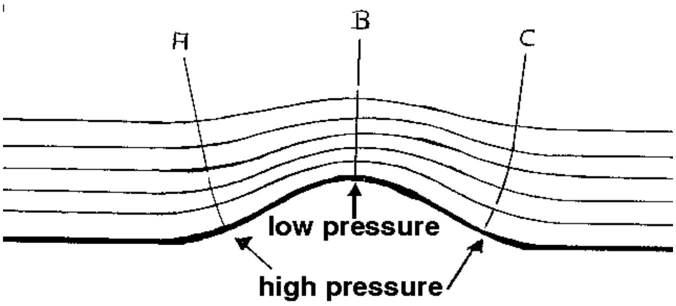

Figure 10

We first analyse the curvature following the trajectory A.The trajectory starts from the bottom of the hill and is continued perpendicular to the streamlines. The streaming air is deflected upwards. The air is accelerated upwards too. Starting from the bottom and going outwards the pressure has to decrease in order to produce the acceleration upwards. Because of the atmospheric pressure further away there must be a higher pressure at the bottom of the trajectory A.

In the case of trajectory B starting from the top of the hill the curvature of the streamlines is reversed. The streaming air is accelerated downwards throughout the whole trajectory. Following this trajectory the pressure increases starting from the top until it reaches its normal value further away. Thus at the top of the hill we expect a reduced pressure. In the case of the trajectory C we expect the same as for trajectory A. Modelling the hill with bent postcards these results can be demonstrated experimentally as well.

Evaporator: These considerations give an explanation of the mechanism for the evaporator. A pipe dipping in a flow of air forces an evasive flow (see figure 11). This is a situation similar to that of the hill. The streaming is curved over the nozzle of the pipe and the acceleration directs to the aperture. Therefore lower pressure is generated at the nozzle.

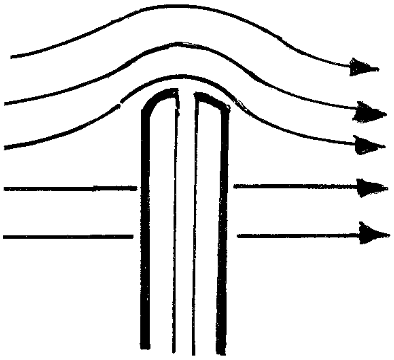

Figure 11

Forces on a roof: If wind passes a house the stream is to flow around it. Due to the curvature of the evasive flow there is higher pressure at the front side and lower pressure at the peak of the roof (figure 12).

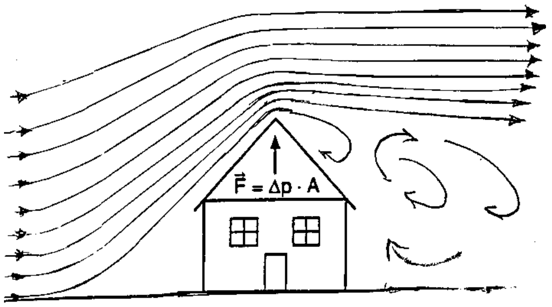

Figure 12

The flow is by no means smooth and laminar. At the peak of the roof it definitely becomes turbulent and separates. (Thus at the rear side we cannot expect the same as for the hill.) Behind the peak of the roof the same reduced pressure can be found as at the peak. This is why the situation at the rear side of the house cannot be the same as for the hill.

The effect of pressure differences on the roof is maximized if front doors or windows are opened. In this case there is high pressure inside the house. The pressure difference acting on the roof is increased.

If windows or doors at the rear are opened there is a lower pressure inside the house that reduces the pressure difference acting on the roof.

Aerodynamic lift: The aerodynamic lift, too, is a result of the evasive flow caused by the aerofoil. The streamlines near the wing are determined by the latter's shape and position. As a whole the stream is deflected downwards. (See figure 8 and figure 9.) For more details of this most important subject see Weltner (2010) and Tipler and Moses(2007)

Propulsion by a sail: The same phenomenon can be observed in the case of a sail. A sail is a curved plane similar to figure 8 and 9. The sail deflects the air flow and produces an increase of pressure at the inner side in relation to the center of curvature and a decrease of pressure at the outer side. By this way it generates a force normal to the sail. Skilled sailors keep the streaming of the air smooth and laminar and avoid turbulent and separating flow.

6. Conclusion

The derivation of Bernoulli's law in schools and textbooks has serious drawbacks. Unfortunately many applications are erranous and misleading. The main source of confusion is the derivation of Bernoulli's law based on the theorem of energy conservation. Bernoulli's law should be derived from the tangential acceleration as a consequence of declining pressure. Another source of difficulties is the fact that many physics textbooks do not mention normal acceleration and the resulting pressure gradients perpendicular to the flow.

Both, Bernoulli's law and the generation of pressure gradients perpendicular to the flow are consequences of Newton's laws. None of them contradicts those.

Bernoulli's law is insufficient to explain the generation of low pressure. A faster streaming velocity never produces or causes lower pressure. The physical cause of low or high pressure is the forced normal acceleration of streaming air caused by obstacles or curved planes in combination with the Coanda-effect. Pressure gradients generated by the deflection of streaming air can be clearly demonstrated by simple experiments which would substantially improve the discussion of fluid mechanics in schools and textbooks.

Literature

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Received 2011-01-06

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Summary: Bernoulli's law and experiments attributed to it are fascinating. Unfortunately some of these experiments are explained erraneously, e.g.: the function of a vaporizer and the soaring of a ping-pong ball in a jet stream of a hair dryer can not be used as applications of Bernoulli's law. The static pressure in a free jet stream is equal to the static pressure in the environmental atmosphere regardless of the streaming velocity of the jet. This can be shown by classroom experiments.

Acceleration of air is caused by pressure gradients. Air is accelerated in direction of the velocity if the pressure goes down. Thus the decrease of pressure is the cause of a higher velocity. It is wrong to say that a lower pressure is caused by a higher velocity.

Pressure gradients perpendicular to the streamlines are caused by the deflection of streaming air. The deflection of air generates regions of lower and higher pressure according to the curvature of the streamlines. Vaporizer, the soaring ping-pong ball as well as the physics of flight are only to be explained regarding the acceleration perpendicular to the streamlines.